Towards a Full-Bandwidth Numerical Acoustic Model

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Abstract
Prediction models are at the heart of modern acoustic engineering and are used in a diverse range of applications from refining the acoustic design of classrooms and concert halls to predicting how noise exposure varies through an urban environment. They also allow Auralisation to be performed for buildings and spaces before they are built or long after they are lost. Current commercial room acoustic simulation software almost exclusively approximates the propagation of sound geometrically as rays or beams. These assumptions yield efficient algorithms, but the maximum accuracy they can achieve is limited by how well the geometric assumption represents sound propagation in a given space. This compromises their accuracy at low frequencies in particular. Methods that directly model wave effects are more accurate but they have a computational cost that scales with problem size and frequency, effectively limiting them to small or low frequency scenarios. This paper will report the results of ongoing research into a new algorithm which aims to be accurate and efficient for all frequencies; the name proposed for this is the “Wave Matching Method”. This builds on the Boundary Element Method with the premise that if an appropriate interpolation scheme is designed then the model will become ‘geometrically dominated’ at high frequencies. Other propagation modes may then be removed without significant error, yielding an algorithm which is accurate and efficient. This paper will present the general concepts of the algorithm and progress to date.

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1. Introduction
Room acoustic simulation is currently split into two distinct camps. Geometrical models of wave propagation are efficient and valid at high frequencies and are widely applied. Wave-based methods such as the Boundary Element Method (BEM) are extremely accurate; however their computational cost scales badly with frequency meaning there is an upper limit to their practical use. Hybrid methods to cover the full audible bandwidth have been proposed [1], however these are essentially two separate methods operating in complementary frequency bands with their results combined in an automated way.

It has also been suggested that a third middle frequency range exists, where neither class of standard approaches performs well and a third type of method is required where some inclusion of the known wave physics is incorporated into a deterministic model [2]. The work in this paper follows along similar lines, but asks if it is possible to set up the wave model in such a way that it resembles the geometric approach at high frequencies. Thus the end goal is not to create a third class of algorithm which must be run in a hybrid scheme, but to create a unified algorithm which is capable of modelling all frequencies with equal efficiency.

Unlike many of the recent ‘middle frequency range’ algorithms, which discretise the wave in the air volume, this work begins from BEM. BEM has a natural synergy with geometric methods since both classes of algorithm work with surface geometry and analytically compute how elementary sound sources propagate through the media unobstructed. The simulation therefore becomes a problem of computing the reflections and scattering from obstacles. Here, in place of the piecewise polynomials usually used in BEM, oscillatory interpolation functions with defined
geometric radiation directions will be used. The idea of using oscillatory interpolation functions in BEM is not new, so a brief review is given in section 2.2 along with a description of how the reported scheme differs. This is followed by a more detailed description of the new algorithm plus some test case results in section 3. Priorities for further work are outlined in section 4.

2. Background

2.1. Standard BEM

In this paper scattering by a rigid obstacle will be considered in the frequency domain. Total pressure $\varphi(x)$ in the medium is a sum of the incident pressure $\chi(x)$ arriving from sources and the scattered pressure $\psi(x)$ which emanates out from the obstacle in response to this. This scattered pressure $\psi(x)$, occurring due to a total pressure $\varphi(y)$ on the surface of the obstacle $\Gamma$, may be calculated using the Kirchhoff-Helmholtz integral equation:

$$\psi(x) = \iint_{\Gamma} \varphi(y) \frac{\partial G}{\partial n}(x,y) d\Gamma_y$$

(1)

Here $G(x,y) = e^{ikR}/4\pi R$ is the free-space Green’s function, where $k$ is the wavenumber and $R = |x - y|$ is the distance from point $y$ to point $x$. $\partial G / \partial n$ means the spatial derivative of $G$ taken in the direction of the surface normal at point $y$.

Note that a term involving $\frac{1}{2} \varphi(x)$ often also appears in equation 1. It is omitted here because the method used to evaluate the surface integral includes that term inside the integral.

In order to permit numerical solution, the surface pressure $\varphi(y)$ is approximated by a sum of basis functions $f_m(y)$ weighted by coefficients $w_n$. In standard BEM the surface to be modelled is partitioned into many elements that are small with respect to wavelength $\lambda$, and the basis functions are defined to be equal to a low-order polynomial on a given element and zero elsewhere. For this to be accurate, a usual rule of thumb is that the elements should be no larger than $\lambda/8$ in each direction. This scaling means that the area of the elements is $O(\lambda^2)$, hence the number of them required to cover the surface is $O(f^2)$. Within each element the surface pressure is interpolated by the set of polynomial functions, each with its own weight. Since the number of these is fixed by the order of polynomial interpolation, the total number of degrees of freedom $N$ of the scheme also scales $O(f^2)$.

The combined set of all the basis functions used may be referred to as the approximation space of the algorithm, since the choice of basis functions defines what space of functions on the surface it can represent well. For example piecewise-constant basis functions will produce a stepped approximation with lots of discontinuities, whereas use of piecewise-linear basis functions will generally produce one which is continuous but not smooth. Later in the paper the discussion will refer to the quality of an approximation space, that is whether it does a good job of representing the true solution $\varphi(y)$.

When the correct distribution of pressure $\varphi(y)$ on $\Gamma$ has been found, then the total pressure $\varphi(x) = \chi(x) + \psi(x)$ predicted by the scheme will be zero if $x$ is in the interior of the obstacle. In order to solve for the set of $w_n$, this assertion must be ‘tested’ at least $N$ times. This could either be done at a finite set of points or in a spatial average sense, and these two options lead to different choices of basis functions. In both cases testing is performed at $x$ locations limitingly close to $\Gamma$, to take advantage of the singularity in $G$ and create a system of equations which is diagonally dominated. The testing points $x$ therefore lie on a surface $\Gamma_\ast$, which is conformal and limitingly close to $\Gamma$ but still inside it. Both schemes produce a matrix equation $Aw = b$, where $b$ (size $N \times 1$) is an excitation term involving $\chi$, $w$ (size $N \times 1$) is the vector of $w_n$, and $A$ (size $N \times N$) is called the interaction matrix. Constructing $A$ is usually the most expensive step in the algorithm.

A scheme which tests the assertion $\chi(x) + \psi(x) = 0$ at a finite set of test points is referred to as a collocation approach. In this case a matching set of test points $x_m$ (called collocation points) and basis functions are chosen such that $f_m(x_m)$ is non-zero only when $m = n$; this leads to the classical BEM shape functions. In contrast in a Galerkin scheme, the assertion is tested in a weak form by multiplying both terms by the conjugate $f_m^*(x)$ of each basis function and then integrating over the surface. The downside of this is that computing the interaction coefficients $A_{m,n}$ for a Galerkin scheme involves evaluating a double surface integral. This incurs a higher computational cost, but the payback is that the method has better error convergence properties [3]. Because of this testing integral, it is desirable that the surface integral over $\Gamma$ of $f_m(x)f_n^*(x)$ is non-zero only when $m = n$. This leads to the use of orthogonal Legendre polynomials to construct Galerkin scheme basis functions.
2.2. BEM with oscillatory basis functions

As stated above, standard BEM uses $O(f^2)$ degrees of freedom to model the oscillations in pressure across the surface of the obstacle, resulting in an interaction matrix $A$ with $O(f^4)$ entries. This makes the method extremely computation and storage intensive at high frequencies. However, when wavelength is small compared to the features of the scattering obstacle then the oscillations occur in directions which are (in principle) predictable using a geometric model of wave propagation. The methods described below attempt to build this oscillation into the basis functions such that only a small number of degrees of freedom are required. In terms of function spaces, this amounts to trying to design an approximation space which is smaller (i.e. has fewer degrees of freedom), but which includes the dominant wave propagation modes so can still do a good job of representing the true solution.

In the following sections three variants of BEM will be discussed. The basis functions follow a similar form in all three, being standard piecewise-polynomial basis functions multiplied by oscillatory functions with the form $e^{iP_n(x)}$. Here $P_n(x)$ are a set of ‘phase functions’, which are chosen to try and capture the high frequency (asymptotic) behaviour. This leaves the standard BEM polynomial functions interpolating only the envelope of the oscillation, and this is (ideally) slowly varying in space so a coarse mesh of elements which are large with respect to wavelength may be used. How the phase functions are chosen, in addition to the choice between collocation or Galerkin schemes, leads to the main differences between the three approaches.

It should be noted however that there is a down side to these oscillatory basis functions. Because each element is now large with respect to wavelength, the computation cost of evaluating each coefficient in the interaction matrix is no longer $O(1)$ but scales with $O(f^2)$ for a collocation scheme and $O(f^4)$ for a Galerkin scheme, if standard numerical integration methods are used. Development of efficient integration methods is an area of active research, and a review of recent progress is given in section 4 of [3].

2.2.1. Partition-of-Unity BEM

The Partition-of-Unity BEM [4] (PU-BEM) is arguably the most established of the mid-frequency BEM algorithms and has been published in two and three-dimensional variants. In this the phase functions represent plane waves propagating in a set of directions. This set of directions is usually chosen to be uniformly spaced in angle, which is straightforward in 2D but less simple in 3D [5]. Dominant wave directions predicted by geometric methods are not generally given special consideration, though the set of basis function wave directions is sometimes rotated so that one of them lines up with the incident wave [6] and the possibility of clustering wave directions close to known dominant diffraction directions has also been suggested [5].

PU-BEM is usually formulated as a collocation scheme and uses classical polynomial shape functions, though a two-dimensional Galerkin scheme has also been reported [7] and design of shape functions with continuous first derivatives is discussed in [8]. Placement of the collocation points appears to be an open question however since, due to the inclusion of the set of wave directions, far more collocation points are required than exist zeros in the shape functions; this is also discussed in [8]. Most PU-BEM algorithms report very poor conditioning of the interaction matrix; this is perhaps no surprise since the choice of basis functions and collocation points has no obvious orthogonality property to provide the matrix with a diagonally dominant structure.

One of the main benefits of PU-BEM is that the method can be readily applied, in so far as it is compatible with standard BEM meshing approaches and curved obstacles [6] and defining the set of phase functions is straightforward. As regards the number of degrees of freedom required to obtain satisfactory accuracy, researchers report that the method generally reduces this from around 10 degrees of freedom per wavelength to 2.5 degrees of freedom per wavelength [4]. This is a significant achievement, but is still $O(f^2)$ so is a step removed from the frequency independent performance of geometric methods.

2.2.2. Hybrid Numerical Asymptotic BEM

Another BEM variant which uses oscillatory basis functions is the Hybrid Numerical Asymptotic BEM (HNA-BEM) [3]. This has sometimes been called ‘Hybrid BEM’ in the literature (including in the title of the research grant which has funded this work), however the term ‘Hybrid Numerical Asymptotic’ will be used here in preference to just ‘Hybrid’ since this is preferred by the mathematical community (from which the work originates) and distinguishes it from different meanings of the word ‘hybrid’ in recently published acoustic modelling algorithms [1].
HNA-BEM differs from the other approaches described here, in that it involves designing a bespoke approximation space for a particular combination of obstacle geometry and source location. Phase functions are chosen to represent specific leading-order diffraction directions for the scattering body under study, and the high-frequency Kirchhoff approximation to the surface quantity (two times the incident field when the surface faces the source) is often subtracted off leaving the BEM scheme to solve only for the remainder. The algorithm is formulated as a Galerkin scheme so Legendre polynomials are used for envelope interpolation and element meshes are often graded to optimise error convergence.

Because of the complications of identifying the leading order diffraction directions, work to date has mostly focussed on specific geometries and is often undertaken in two dimensions. For example, the diffraction from a convex polygonal obstacle [9] is well suited to the HNA-BEM approach in two dimensions, since the wave directions are always surface tangential and propagate either clockwise or anti-clockwise, but the best way to design the approximation space for a three-dimensional polyhedral obstacle is unresolved. It also means that the choice of phase functions is often dependent on the source position, which could be a significant computational overhead if many source positions are required. However, for the cases where the approach can be applied it has achieved extremely impressive results, for example reducing the number of degrees of freedom from $O(f)$ to $O(\log f)$ for a convex polygonal obstacle in two dimensions [9] and from $O(f^2)$ to $O(1)$ for a sphere [10].

2.2.3. ‘Wave-Matching’ method

The ‘Wave-Matching’ algorithm [11] takes another slightly different approach, this time not only inferring that the solution will become geometrically dominated at high frequencies, but also that the interaction between pairs of wave terms will as well. The concept is based around the idea of a surface reflectance model, where waves arrive from the medium, are processed by a model of the obstacle according to its material properties, and then the reflected waves (if any) are radiated back into the medium. This couples together the discretisation of the surface pressure and surface-normal particle velocity into a set of incoming waves and a set of outgoing waves. The terms used to discretise the surface quantities are therefore defined not only on the surface but also as waves in the medium. If these basis functions satisfy the wave equation then it can be shown (using a transform developed for the problem of aperture diffraction [12,13,14]) that their scattering can be stated as a geometric term plus an edge-diffraction contour integral. Since the contour integrals are 1D it follows that they can be evaluated in $O(f)$ operations, so scale better than standard quadrature methods. In addition, the increased validity of the geometric approximation at high-frequencies suggests that the contribution of the diffraction integral must become relatively insignificant at high frequencies.

To date the wave functions considered have only been plane waves, hence the basis functions still fit the ansatz given at the start of section 2.2. However the requirement that the basis functions must be solutions of the wave equation windowed to some finite support on the surface restricts the choice of shape functions to piecewise-constant only. The method is formulated as a Galerkin scheme, so it is beneficial if the basis functions are chosen such that they are orthogonal over the surface of the obstacle. For an obstacle with a surface composed of rectangular faces an obvious choice is to design the wave directions such that they form a 2D Fourier series. In terms of wave directions, the component of wavenumber perpendicular to the face is set such that the total wavenumber in three dimensions matches $k$ in the medium; this is commonly known as a ‘wavenumber spectrum’ [15].

Such a representation uses many more wave directions than either PU-BEM or HNA-BEM. In fact it is superficially no better than an elements scheme since it still requires $O(f^2)$ terms to adequately represent all possible propagation modes. However in practice it appears that many of these terms are not significantly excited and the resulting interaction matrix is approximately sparse [11] (i.e. the vast majority of coefficients are extremely small compared to a small number of very large magnitude coefficients). It is therefore intended that the approximation space of basis functions proposed here be regarded as a coordinate space for indexing a large number of pre-defined wave directions, of which very few will actually be used. Similar sparsity has been reported in some HNA-BEM implementations [16] and occurs because: 1) at high wavenumbers the scattered field becomes beam-like and only propagates significantly in certain directions, and 2) the phase functions of an incoming wave and a testing basis function must match exactly in order.
for the oscillations to demodulate and avoid cancellation inside the testing integral. The primary advantage of this approach over HNA-BEM is that the definition of the basis functions is independent of the excitation wave, so changing the source position would not demand total recalculation of the interaction matrix.

3. Wave-Matching test case algorithm

A test case algorithm using the Wave-Matching will now be reported. This builds on the results reported in reference [11], with the primary difference that the interaction matrices were constructed approximately there by transforming the interaction matrices of a regular BEM, whereas they are integrated exactly herein using the method given in [13].

The surface \(\Gamma\) bounding the obstacle is comprised of planar rectangular faces. Each face \(\Gamma_a\) is defined by a corner vector \(v_a\) and two perpendicular edge vectors \(e_{a;1}\) and \(e_{a;2}\), with normal vector \(\hat{n}_a = \hat{e}_{a;1} \times \hat{e}_{a;2}\). On each face a two-dimensional Fourier series is used to approximate the pressure. This is particularly suitable for rectangular faces since the basis functions are exactly orthogonal; they are defined:

\[
f_{a,m,n}(y) \begin{cases} e^{i(\alpha y - v_a)k_{a,m,n}} & \text{if } y \in \Gamma_a \\ 0 & \text{otherwise} \end{cases}
\]  

Here \(m\) and \(n\) are spatial harmonic indexes and the wavenumber \(k_{a,m,n}\) is defined as:

\[
k_{a,m,n} \equiv \frac{2\pi m}{|e_{a;1}|} \hat{e}_{a;1} + \frac{2\pi n}{|e_{a;2}|} \hat{e}_{a;2}
\]  

The total pressure \(\varphi(y)\) on the surface is approximated by a weighted sum of these:

\[
\varphi(y) \approx \sum_a \sum_m \sum_n w_{a,m,n} f_{a,m,n}(y)
\]  

The limits in \(m\) and \(n\) are chosen to be odd, so as to ensure a symmetric wavenumber spectrum, and such that they give the equivalent of eight degrees of freedom per wavelength. Ultimately the three summation indices are mapped into one index so that a standard matrix equation as discussed in section 2.1 is constructed.

It should be noted that each \(f_{a,m,n}(y)\) really represents the sum of two waves \(f_{a,m,n}^\pm\) which are mirror images in the plane of \(\Gamma_a\); one incoming \((-\)\) and one outgoing \((+\)\) with respect to the obstacle. These are members of two symmetrically opposed wavenumber spectrums, where the non-surface tangential component of wavenumber has been taken to either lie in the direction \(-\hat{n}_a\) or \(+\hat{n}_a\) respectively. They have wavenumbers:

\[
k_{a,m,n}^\pm \equiv k_{a,m,n} \pm i \sqrt{k^2 - \left(\frac{2\pi m}{|e_{a;1}|}\right)^2 - \left(\frac{2\pi n}{|e_{a;2}|}\right)^2}
\]

Note that for high values of \(m\) and \(n\) \((2\pi m/|e_{a;1}|)^2 + (2\pi n/|e_{a;2}|)^2\) may be larger than \(k^2\), so the square root will produce an imaginary number. This corresponds to the wave being evanescent.

Since each face is planar and rigid the reflection coefficient between each pair of waves is always unity. They have therefore been collapsed into one basis function in equation 2 to simplify the algorithm and halve the number of degrees of freedom; a similar approach was successfully used in the time domain for surface-normal plane waves [17]. However their definition is important because the integral transform used [13] requires that the basis function be a windowed plane wave. Despite having been known for such a long time, it appears that this technique has not been applied to BEM integrals before, to the best of the authors’ knowledge. Interestingly they can be shown to be equivalent to the well-known piecewise-constant polar integrals [18] for the \(m = n = 0\) case.

The above is how the ‘scattering’ integral in equation 1 is calculated, but since the formulation is a Galerkin scheme it also includes ‘testing’ integrals involving the conjugate of another basis function. This could be computed using standard quadrature rules (e.g. Gauss-Legendre), however use of regularly-spaced abscissa allows the testing integral to be computed simultaneously for all wave directions in the scheme using a two-dimensional fast Fourier transform as outlined in Equation 24 of reference [11].

3.1. Results with full interaction matrices

As discussed in section 2.2.3, the concept behind this algorithm is that not all basis functions defined in the approximation space will be required, and not all coefficients in the interaction matrix will be computed. However development of an accurate method to identify in advance which coefficients to compute is ongoing, so here the full matrix will be computed first and then the smaller magnitude coefficients will be discarded according to a process described in section 3.3. In this section however, error data is computed with the full interaction matrix to give a baseline for the
accuracy which can be achieved with the algorithm. The results of the algorithm under test are compared to a standard BEM solver using 10 piecewise constant elements per wavelength. The error data given is the $L_2$ norm of the error between the two methods, computed over the surface of the obstacle, divided by the $L_2$ norm of the reference solution, to give a unit-less normalised error which is expressed in percent. In all cases the scattering obstacle was a 1m cube and the speed of sound was taken to be 343m/s.

Figure 1 shows the normalised error data versus frequency for a point source located at (0,0,10) and at (10,20,30). The error is relatively high (for a BEM algorithm) in both cases, peaking at around 8%-14%, though it does appear to reduce with increasing frequency. This poses some important questions regarding the suitability of the approximation space given in equation 4, and these will be addressed in the next section.

Another aspect which should be considered is the condition number of the interaction matrix $A$. Very high condition numbers are common in other schemes which use plane waves as basis functions; for example values of the order $10^{15}$ are not uncommon for PU-BEM [19], necessitating the use of special matrix solvers. In contrast, the choice of phase functions used here meant that all condition numbers in the above simulations were well below $10^2$.

Consider a one dimensional version of the scheme described above. $\varphi(x)$ is the quantity to approximate on a domain from $x = 0$ to $x = L$. The approximation is done using a Fourier series, which is effectively a weighted sum of basis functions $f_n(x) = e^{i2\pi nx/L}$:

$$\varphi(x) = \sum_{n=-\infty}^{\infty} \Phi_n f_n(x)$$ (6)

The question is how well can this represent a plane wave $\varphi(x) = e^{ikx}$ whose wavenumber $k$ is not an integer multiple of $2\pi/L$. In particular, the representation used in equation 4 is of course not an infinite sum like this but a finite sum between limits $\pm[N/2]$. The summation in equation 6 is therefore truncated, leading to a truncation error:

$$E(x) = \sum_{n=[N/2]}^{\infty} [\Phi_n f_n(x) + \Phi_{-n} f_{-n}(x)]$$ (7)

To understand the consequences of this it is illustrative to look at the values of $\Phi_n$ for various values of $k$. As is well known, the Fourier transform of a complex exponential times a rectangular window is a sinc function. The values of $\Phi_n$ amount to evaluating this at integer $n$:

$$\Phi_n = \frac{1}{L} \int_0^L \varphi(x) f_n^*(x) dx$$

$$= e^{i[kL-2\pi n]/2} \text{sinc}(\lfloor kL - 2\pi n \rfloor/2)$$ (8)

Figures 2 and 3 show the results of this for two different values of $k$. In figure 2, $k$ is chosen such that $\varphi(x)$ exactly matches one basis function. As is well known, this causes the sinc function in $n$ to oscillate such that it crosses zero at every integer $n$, leaving only one non-zero Fourier series coefficient. In contrast figure 3 shows $\Phi_n$ when $k$ is chosen such that $\varphi(x)$ doesn’t exactly match any basis function. In this case the oscillations of the sinc function are offset and a great many $\Phi_n$ coefficients are non-zero. In addition, the magnitude of the $\Phi_n$ coefficients decays quite slowly away from the peak value; this is bad news because it means the truncation error given in equation 7 is likely to be quite significant and that
the approximate solution given by the finite sum will converge slowly to the correct result as the value of \( N \) is increased.

![Figure 2: Fourier series coefficients (dots) and the continuous Fourier transform (line) of \( \varphi(x) = e^{ikx} \) over the domain \( x = 0 \) to \( x = L \) where \( k = \frac{3.0 \times 2\pi}{L} \)](image)

![Figure 3: Fourier series coefficients (dots) and the continuous Fourier transform (line) of \( \varphi(x) = e^{ikx} \) over the domain \( x = 0 \) to \( x = L \) where \( k = \frac{3.5 \times 2\pi}{L} \)](image)

Figure 2 shows how the normalised truncation error varies with the value of \( k \) for \( N = 33 \). It can be seen that the error oscillates and is zero whenever \( k \) is an integer multiple of \( 2\pi/L \). It can also be seen that error increases when \( k \) is close to the truncation limits of \( \pm \lceil N/2 \rceil \) at the ends of the plotting range. However, the typical choice of \( N \) ensures that \( k \) will usually lie in the centre of this range as indicated by the red box. The maximum error of the algorithm is therefore expected to be around 11.3%, which matches well with the error magnitudes observed in figure 1. The oscillation in error seen there in the face-normal source case can now also be explained. In this case, the incident wave is approximately equal to a plane wave arriving perpendicular to one face. On the side faces the dominant field is therefore a surface-tangential plane wave with the same form as the one-dimensional \( \varphi(x) \) used in this section. The two frequencies where the error is lowest (350Hz and 700Hz) are those that are closest to the two frequencies (343Hz and 686Hz) at which this analysis predicts error should be minimal.

![Figure 4: \( L_2 \) norm of the truncation error \( E(x) \) normalised by the \( L_2 \) norm of \( \varphi(x) \) expressed in percent versus a scaled version of \( k \). The red box indicates the range of \( k \) used in the algorithm (for this \( N \) and the corresponding expected error limit.)](image)

3.3. Interaction matrix coefficient culling

The effect of ‘culling’ small coefficients from the interaction matrix will be investigated here because this is an essential step in demonstrating the potential efficiency of the method. Without sparsity the scheme uses the same number of degrees of freedom and has the same size interaction matrices as a conventional BEM, but with more expensive integration!

As has previously been reported [11, 20], the interaction matrix \( A \) which arises from the scheme outlined above is already approximately sparse; i.e. it has a small number of large magnitude entries and a very large number of small magnitude entries. Figure 5 shows these statistics in the form of a frequency dependent histogram. The horizontal scale gives coefficient magnitude normalised to the largest magnitude entry at that frequency. Colour indicates the proportion of coefficients in the matrix in a certain magnitude ‘bin’, again normalised for each frequency so that black indicates the bins with the most coefficients and white represents those with the fewest. The trend is that the vast majority of the coefficients in the matrix are between \( 10^{-2} \) and \( 10^{-4} \) times smaller than the largest coefficient, and the largest coefficients (right-hand limit of the plot) are so few that this area appears white. Development of an accurate method to identify in advance which coefficients will be significant (so only those need be computed) is ongoing area of research. However numerical experiments have shown that the very largest of the coefficients can be
identified by considering which waves line up geometrically. It that sense there are some parallels between this approach and the ray direction quantisation work of Pohl [21].

Figure 5: Histogram showing normalised interaction coefficient magnitude statistics versus frequency.

Because BEM relies on cancellation and interference between different terms, it is unwise to use a hard cut-off to cull interactions and instead a soft cut-off was devised. This operated by applying a linear taper between two thresholds and is described in detail in reference [11].

To try and quantify the effects of the culling, the threshold was gradually increased, starting with a very small value which left the matrix unaffected up to a value which affected all the coefficients. For each value of the threshold the normalised $L_2$ error in the solution was calculated for the oblique source position, along with the condition number of the culled interaction matrix and the percentage population (the number of non-zeros in $A$ divided by its total number of elements). Figure 6 show an example of the error trend observed at a frequency of 850Hz. Here the error appears to be unaffected by the culling until it reaches a point where at least 90% of the coefficients in the interaction matrix have been discarded, at which point it rises sharply. This implies that (as hoped) the vast majority of the interaction coefficients (the dark band in figure 5) may be set to zero without significantly affecting the solution produced by the algorithm. The condition number followed a similar trend, being largely unaffected until the largest magnitude coefficients are altered.

Figure 6: Normalised $L_2$ error in the solution versus percentage population in the culled interaction matrix at 850Hz.

Figure 7: Number of coefficients which must be retained in $A$ to ensure normalised error is below 15%.

Figure 8: number of coefficients which must be retained in $w$ to ensure normalised error is below 15%.
The same process was repeated for a range of frequencies and the percentage population required to keep the normalised error below a certain percentage was calculated in each case. This error threshold was chosen to be 15%, which is quite high but it was necessary that it was above the base error reported in section 3.1. The results of this are shown in figure 7. Note that the figure includes two vertical axes, and the left scale is 20 times the right scale. The solid blue line (left vertical axis) shows the total number of elements in the interaction matrix. As discussed above this scales $O(f^4)$, and the dashed blue line is an $f^4$ best fit trend. The solid green line (right vertical axis) shows the number of coefficients which must be retained in order to ensure the normalised $L_2$ error remains below 15%. Again a dashed line of best fit is included, and in this case a line which follows an $f^3$ trend gives the closest match. It therefore appears from this preliminary data that the number of interaction coefficients required by the scheme has been reduced from $O(f^4)$ to $O(f^3)$. It should be reiterated though that a method to identify these terms in advance is required to make the scheme efficient.

Finally, the same culling process was applied to the $w$ vector of discretisation coefficients to establish how the number of basis functions required scales with frequency. The results of this are shown in figure 8 and again this includes two vertical axes, where the left scale is 10 times the right scale. The solid blue line (left vertical axis) shows the total number of degrees of freedom referenced by the discretisation scheme. As discussed previously this scales $O(f^2)$, and the dashed blue line is an $f^2$ line of best fit. The solid green line (right vertical axis) shows the number of coefficients which must be retained in order to ensure the normalised $L_2$ error remains below 15%. Again a dashed line of best fit is included. After an initial rise with $f^2$ this appears to settle at some constant amount plus a number which rises $O(f)$, and the dashed line follows a best fit with $f$ in this region. Again this suggests that a reduction in the scaling of the number of terms required has been achieved.

4. Further work

It is clear from the above results that the use of plane wave functions with piecewise constant envelopes has its benefits as well as its flaws. Elimination of the polynomial envelope has allowed more efficient integration of the inner scattering integral in $O(f)$ operations, however the base error is much higher than has been achieved with schemes that mix plane waves with continuous polynomial envelopes [6].

Another possibility may however arise from the trigonometric envelope functions recently proposed by Peake et al [8], since these may also be decomposed into sums of plane waves. Take their central envelope function $N$ from Equation 22 of reference [8], which is essentially a Hanning window. If defined over the support $x = 0$ to $x = L$ then this is given as:

$$w(x) = \frac{1}{2} - \frac{1}{2} \cos(2\pi x/L) = \frac{1}{4} - \frac{1}{4} e^{i2\pi x/L} - \frac{1}{4} e^{-i2\pi x/L}$$

(9)

Its Fourier series spectrum (over the domain $x = 0$ to $x = L$) is:

$$W_n = \begin{cases} \frac{1}{2} & n = 0 \\ \frac{-1}{4} & n = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

(10)

It follows that the Fourier series transform of $w(x)\varphi(x)$ can readily be found by convolving $\Phi_n$ with the three non-zero values of $W_n$. Since the scattering from each plane wave term can be computed in $O(f)$ operations using the contour integral transform described in section 3, it follows that the scattering from this particular type of windowed basis function may also be computed in $O(f)$ time.

The effect of the windowing on the values of $\Phi_n$ is shown in figures 9 and 10. For the case where $k$ is an exact integer multiple of $2\pi/L$ (figure 9) the situation has worsened marginally; three $\Phi_n$ are now non-zero instead of one. However the continuous $n$ Fourier transform decays much more quickly than it did in figures 2 and 3. This means that when $k$ is not an exact integer multiple of $2\pi/L$ (figure 10) then the situation is much improved and there are still only a small number of coefficients with significant magnitude. This can be understood by considering that the periodic extension of $\varphi(x) = e^{ikx}$ contains a discontinuity when $k$ is not an exact integer multiple of $2\pi/L$, giving rise to Gibbs artefacts, whereas $w(x)\varphi(x)$ is continuous in its $1^{\text{st}}$ and $2^{\text{nd}}$ derivatives regardless of the value of $k$. This faster convergence in $n$ makes a big difference to the truncation error. This was calculated for the same configuration illustrated in figure 4, and windowing brought the maximum expected truncation error down from 11.3% to just 0.005%.
The effect of culling the Fourier series coefficients was also investigated for the same range of $k$. Setting the upper threshold to a value of 0.1 produced a representation which used at most just 6 terms yet produced an maximum error of only 0.68%. It therefore seems likely that such a scheme with trigonometrically windowed basis functions would greatly outperform the scheme described herein, both in terms of accuracy and matrix sparsity.

Figure 9: Fourier series coefficients (dots) and the continuous Fourier transform (line) of $\varphi(x) w(x)$ over the domain $x = 0$ to $x = L$ where $k = 3.0 \times 2\pi / L$

Figure 10: Fourier series coefficients (dots) and the continuous Fourier transform (line) of $\varphi(x) w(x)$ over the domain $x = 0$ to $x = L$ where $k = 3.5 \times 2\pi / L$

Another aspect which is simplified in this paper compared to previous publications is the testing integral operator. The previous two papers describing this method [11,20] used a combination of pressure and surface-normal particle velocity in the testing statement, however here pressure only is used. This simplifies the implementation and side-steps the issue of how to correctly evaluate the hyper-singular operator which would appear in the interaction matrix; Diwan et al [19] recently investigated use of the Burton Miller formulation [22] for PU-BEM & encountered similar problems. Without this however the method is susceptible to the well-known non-uniqueness issue. The CHIEF method [23] is another widely used approach to handle this, however it is not reliable at high frequencies [24] and complicates the design of time domain solvers [25] (creation of the latter being a long-term goal). It therefore seems that implementation of some testing scheme which involves the hyper-singular integral will be necessary, and this is an ongoing area of research.

5. Conclusions

A test-case version of a new variant of BEM called the ‘wave-matching’ method has been presented. This uses windowed solutions of the wave equation as basis functions, leading to an interaction matrix which is dominated by a very small number of significant entries. Similarities and differences to other previously published algorithms were discussed, alongside results quantifying its performance. In particular this algorithm chooses its wave directions such that they form a 2D Fourier series on the faces of the obstacle, resulting in very good condition numbers compared to other similar algorithms. In the present scheme the base error is however quite high, at around 8-14%, and this is explained in terms of truncation of the infinite series of terms in the Fourier series. Nonetheless it was shown that culling many of the smaller coefficients from the interaction matrix to create a sparse representation does not increase this error, and a solution involving trigonometric windows was proposed which may reduce the base error.

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References


